Surface Area and Volume

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**CHAPTER 1**

**Surface Area and Volume**

**CHAPTER OUTLINE**

1.1 Polyhedrons
1.2 Cross-Sections and Nets
1.3 Prisms
1.4 Cylinders
1.5 Pyramids
1.6 Cones
1.7 Spheres
1.8 Composite Solids
1.9 Area and Volume of Similar Solids

**Introduction**

In this chapter we extend what we know about two-dimensional figures to three-dimensional shapes. First, we will define the different types of 3D shapes and their parts. Then, we will find the surface area and volume of prisms, cylinders, pyramids, cones, and spheres.
Here you’ll learn what a polyhedron is and the parts of a polyhedron. You’ll then use these parts in a formula called Euler’s Theorem.

What if you were given a solid three-dimensional figure, like a carton of ice cream? How could you determine how the faces, vertices, and edges of that figure are related? After completing this Concept, you’ll be able to use Euler’s Theorem to answer that question.

**Watch This**

[Image: Multimedia]

Click image to the left for more content.

Polyhedrons CK-12

**Guidance**

A **polyhedron** is a 3-dimensional figure that is formed by polygons that enclose a region in space. Each polygon in a polyhedron is a *face*. The line segment where two faces intersect is an *edge*. The point of intersection of two edges is a *vertex*.

Examples of polyhedrons include a cube, prism, or pyramid. Cones, spheres, and cylinders are not polyhedrons because they have surfaces that are not polygons. The following are more examples of polyhedrons:
The number of faces \( F \), vertices \( V \) and edges \( E \) are related in the same way for any polyhedron. Their relationship was discovered by the Swiss mathematician Leonhard Euler, and is called Euler’s Theorem.

**Euler’s Theorem:** \( F + V = E + 2 \).

\[
\begin{align*}
\text{Faces} + \text{Vertices} &= \text{Edges} + 2 \\
5 + 6 &= 9 + 2
\end{align*}
\]

A **regular polyhedron** is a polyhedron where all the faces are congruent regular polygons. There are only **five regular polyhedra, called the Platonic solids.**

1. **Regular Tetrahedron:** A 4-faced polyhedron and all the faces are equilateral triangles.
2. **Cube:** A 6-faced polyhedron and all the faces are squares.
3. **Regular Octahedron:** An 8-faced polyhedron and all the faces are equilateral triangles.
4. **Regular Dodecahedron:** A 12-faced polyhedron and all the faces are regular pentagons.
5. **Regular Icosahedron:** A 20-faced polyhedron and all the faces are equilateral triangles.

**Example A**

Determine if the following solids are polyhedrons. If the solid is a polyhedron, name it and find the number of faces, edges and vertices it has.

a)
1.1. Polyhedrons

b) The base is a triangle and all the sides are triangles, so this is a triangular pyramid, which is also known as a **tetrahedron**. There are 4 faces, 6 edges and 4 vertices.

c) This solid is also a polyhedron. The bases are both pentagons, so it is a pentagonal prism. There are 7 faces, 15 edges, and 10 vertices.

c) The bases are circles. Circles are not polygons, so it is not a polyhedron.

**Example B**

Find the number of faces, vertices, and edges in an octagonal prism.

There are 10 faces and 16 vertices. Use Euler’s Theorem, to solve for $E$.  

\[ F + V = E + 2 \]
\[ 10 + 16 = E + 2 \]
\[ 24 = E \]

Therefore, there are 24 edges.

**Example C**

A *truncated icosahedron* is a polyhedron with 12 regular pentagonal faces, 20 regular hexagonal faces, and 90 edges. This icosahedron closely resembles a soccer ball. How many vertices does it have? Explain your reasoning.

We can use Euler’s Theorem to solve for the number of vertices.

\[ F + V = E + 2 \]
\[ 32 + V = 90 + 2 \]
\[ V = 60 \]

Therefore, it has 60 vertices.

**Guided Practice**

1. In a six-faced polyhedron, there are 10 edges. How many vertices does the polyhedron have?
2. Markus counts the edges, faces, and vertices of a polyhedron. He comes up with 10 vertices, 5 faces, and 12 edges. Did he make a mistake?
3. Is this a polyhedron? Explain.
1. Solve for $V$ in Euler’s Theorem.

\[
F + V = E + 2 \\
6 + V = 10 + 2 \\
V = 6
\]

Therefore, there are 6 vertices.

2. Plug all three numbers into Euler’s Theorem.

\[
F + V = E + 2 \\
5 + 10 = 12 + 2 \\
15 \neq 14
\]

Because the two sides are not equal, Markus made a mistake.

3. All of the faces are polygons, so this is a polyhedron. Notice that even though not all of the faces are regular polygons, the number of faces, vertices, and edges still works with Euler’s Theorem.

**Practice**

Complete the table using Euler’s Theorem.

<table>
<thead>
<tr>
<th>Name</th>
<th>Faces</th>
<th>Edges</th>
<th>Vertices</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Rectangular Prism</td>
<td>6</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>2. Octagonal Pyramid</td>
<td>16</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>3. Regular Icosahedron</td>
<td>20</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>4. Cube</td>
<td>12</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>5. Triangular Pyramid</td>
<td>4</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>6. Octahedron</td>
<td>8</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>7. Heptagonal Prism</td>
<td>21</td>
<td>14</td>
<td></td>
</tr>
<tr>
<td>8. Triangular Prism</td>
<td>5</td>
<td>9</td>
<td></td>
</tr>
</tbody>
</table>

Determine if the following figures are polyhedra. If so, name the figure and find the number of faces, edges, and vertices.
Here you’ll learn different ways of representing three-dimensional objects in two dimensions. In particular, you’ll learn about cross-sections and nets.

What if you were given a three-dimensional figure like a pyramid and you wanted to know what that figure would look like in two dimensions? What would a flat slice or an unfolded flat representation of that solid look like? After completing this Concept, you’ll be able to use cross-sections and nets to answer questions like these.

Watch This

Three Dimensions in Two Dimensions CK-12

Guidance

While our world is three dimensional, we are used to modeling and thinking about three dimensional objects on paper (in two dimensions). There are a few common ways to help think about three dimensions in two dimensions. One way to “view” a three-dimensional figure in a two-dimensional plane (like on a piece of paper or a computer screen) is to use cross-sections. Another way to “view” a three-dimensional figure in a two-dimensional plane is to use a net.

**Cross-Section:** The intersection of a plane with a solid.

The cross-section of the peach plane and the tetrahedron is a *triangle*.

**Net:** An unfolded, flat representation of the sides of a three-dimensional shape.

It is good to be able to visualize cross sections and nets as the three dimensional objects they represent.
Example A

What is the shape formed by the intersection of the plane and the regular octahedron?

a) 

b) 

c) 

Answer:

a) Square
b) Rhombus
c) Hexagon

Example B

What kind of figure does this net create?
1.2. Cross-Sections and Nets

The net creates a rectangular prism.

Example C

Draw a net of the right triangular prism below.

The net will have two triangles and three rectangles. The rectangles are different sizes and the two triangles are the same.

There are several different nets of any polyhedron. For example, this net could have the triangles anywhere along the top or bottom of the three rectangles. Click the site http://www.cs.mcgill.ca/~sqrt/unfold/unfolding.html to see a few animations of other nets.
Guided Practice

1. Describe the cross section formed by the intersection of the plane and the solid.

2. Determine what shape is formed by the following net.

3. Determine what shape is formed by the following net.

Answers:
1. A circle.
2. A cube.
3. A square-based pyramid.

Practice

Describe the cross section formed by the intersection of the plane and the solid.
Draw the net for the following solids.

Determine what shape is formed by the following nets.
1.3 Prisms

Here you’ll learn what a prism is and how to find its volume and surface area.

What if you were given a solid three-dimensional figure with two congruent bases in which the other faces were rectangles? How could you determine how much two-dimensional and three-dimensional space that figure occupies? After completing this Concept, you’ll be able to find the surface area and volume of a prism.

Watch This

Guidance

A prism is a 3-dimensional figure with 2 congruent bases, in parallel planes, in which the other faces are rectangles.

The non-base faces are lateral faces. The edges between the lateral faces are lateral edges.

This particular example is a pentagonal prism because its base is a pentagon. Prisms are named by the shape of their base. Prisms are classified as either right prisms (prisms where all the lateral faces are perpendicular to the bases) or oblique prisms (prisms that lean to one side, whose base is a parallelogram rather than a rectangle, and whose height is perpendicular to the base’s plane), as shown below.
Surface Area

To find the **surface area** of a prism, find the sum of the areas of its faces. The **lateral area** is the sum of the areas of the lateral faces. The basic unit of area is the square unit.

\[
\text{Surface Area} = B_1 + B_2 + L_1 + L_2 + L_3 \\
\text{Lateral Area} = L_1 + L_2 + L_3
\]

![Diagram showing surface area and lateral area of a prism]

Volume

To find the **volume** of any solid you must figure out how much space it occupies. The basic unit of volume is the cubic unit.

For prisms in particular, to find the volume you must find the area of the base and multiply it by the height.

**Volume of a Prism:** \( V = B \cdot h \), where \( B \) = area of base.

![Diagram showing volume of a prism]

If an oblique prism and a right prism have the same base area and height, then they will have the same volume. This is due to **Cavalieri’s Principle**, which states that if two solids have the same height and the same cross-sectional area at every level, then they will have the same volume.

Example A

Find the surface area of the prism below.
To solve, draw the net of the prism so that we can make sure we find the area of ALL faces. Using the net, we have:

\[
SA_{\text{prism}} = 2(4)(10) + 2(10)(17) + 2(17)(4) \\
= 80 + 340 + 136 \\
= 556 \text{ cm}^2
\]

**Example B**

Find the surface area of the prism below.

This is a right triangular prism. To find the surface area, we need to find the length of the hypotenuse of the base because it is the width of one of the lateral faces. We can use the Pythagorean Theorem to find this length.
\[7^2 + 24^2 = c^2\]
\[49 + 576 = c^2\]
\[625 = c^2 \quad c = 25\]

Looking at the net, the surface area is:

\[SA = 28(7) + 28(24) + 28(25) + 2 \left( \frac{1}{2} \cdot 7 \cdot 24 \right)\]
\[SA = 196 + 672 + 700 + 168 = 1736 \, \text{units}^2\]

**Example C**

You have a small, triangular prism-shaped tent. How much volume does it have once it is set up?

First, we need to find the area of the base.

\[B = \frac{1}{2}(3)(4) = 6 \, \text{ft}^2\]
\[V = Bh = 6(7) = 42 \, \text{ft}^3\]

Even though the height in this problem does not look like a “height,” it is because it is the perpendicular segment connecting the two bases.

**Guided Practice**

1. The total surface area of the triangular prism is 540 \(\text{units}^2\). What is \(x\)?
2. Find the volume of the right rectangular prism below.

3. A typical shoe box is 8 in by 14 in by 6 in. What is the volume of the box?

**Answers:**

1. The total surface area is equal to:

\[ A_{\text{2 triangles}} + A_{\text{3 rectangles}} = 540 \]

The hypotenuse of the triangle bases is 13, \( \sqrt{5^2 + 12^2} \). Let’s fill in what we know.

\[ A_{\text{2 triangles}} = 2 \left( \frac{1}{2} \cdot 5 \cdot 12 \right) = 60 \]
\[ A_{\text{3 rectangles}} = 5x + 12x + 13x = 30x \]
\[ 60 + 30x = 540 \]
\[ 30x = 480 \]
\[ x = 16 \text{ units} \quad \text{The height is 16 units.} \]

2. The area of the base is \( (5)(4) = 20 \) and the height is 3. So the total volume is \( (20)(3) = 60 \text{ units}^3 \)

3. We can assume that a shoe box is a rectangular prism.

\[ V = (8)(14)(6) = 672 \text{ in}^3 \]
Interactive Practice

Practice

1. What type of prism is this?

2. Draw the net of this prism.
3. Find the area of the bases.
4. Find the area of lateral faces, or the lateral surface area.
5. Find the total surface area of the prism.

6. How many one-inch cubes can fit into a box that is 8 inches wide, 10 inches long, and 12 inches tall? Is this the same as the volume of the box?
7. A cereal box in 2 inches wide, 10 inches long and 14 inches tall. How much cereal does the box hold?
8. A can of soda is 4 inches tall and has a diameter of 2 inches. How much soda does the can hold? Round your answer to the nearest hundredth.
9. A cube holds 216 $in^3$. What is the length of each edge?
10. A cube has sides that are 8 inches. What is the volume?

Use the right triangular prism to answer questions 11-15.

11. Find the volume of the prism.
12. What shape are the bases of this prism? What are their areas?
13. What are the dimensions of each of the lateral faces? What are their areas?
14. Find the lateral surface area of the prism.
15. Find the total surface area of the prism.
16. Describe the difference between lateral surface area and total surface area.
17. Fuzzy dice are cubes with 4 inch sides.
a. What is the volume and surface area of one die?
b. What is the volume and surface area of both dice?

Find the volume of the following solids. Round your answers to the nearest hundredth.

18. bases are isosceles trapezoids

Find the value of $x$, given the surface area.

22. $V = 504 \text{ units}^3$
23. $V = 2688 \text{ units}^3$
1.4 Cylinders

Here you’ll learn what a cylinder is and how to find its volume and surface area.

What if you were given a solid three-dimensional figure with congruent enclosed circular bases that are in parallel planes? How could you determine how much two-dimensional and three-dimensional space that figure occupies? After completing this Concept, you’ll be able to find the surface area and volume of a cylinder.

Watch This

Cylinders CK-12

Guidance

A cylinder is a solid with congruent circular bases that are in parallel planes. The space between the circles is enclosed.

A cylinder has a radius and a height.

A cylinder can also be oblique (slanted) like the one below.
Surface Area

Surface area is the sum of the area of the faces of a solid. The basic unit of area is the square unit.

Surface Area of a Right Cylinder: \( SA = 2\pi r^2 + 2\pi rh \).

\[
\begin{align*}
2\pi r^2 & + 2\pi rh \\
\text{area of circles} & \text{length of rectangle}
\end{align*}
\]


Volume

To find the volume of any solid you must figure out how much space it occupies. The basic unit of volume is the cubic unit. For cylinders, volume is the area of the circular base times the height.

Volume of a Cylinder: \( V = \pi r^2 h \).

If an oblique cylinder has the same base area and height as another cylinder, then it will have the same volume. This is due to Cavalieri’s Principle, which states that if two solids have the same height and the same cross-sectional area at every level, then they will have the same volume.

Example A

Find the surface area of the cylinder.
1.4. Cylinders

$r = 4$ and $h = 12$.

\[
SA = 2\pi(4)^2 + 2\pi(4)(12) \\
= 32\pi + 96\pi \\
= 128\pi \text{ units}^2
\]

**Example B**

The circumference of the base of a cylinder is $16\pi$ and the height is $21$. Find the surface area of the cylinder.

We need to solve for the radius, using the circumference.

\[
2\pi r = 16\pi \\
r = 8
\]

Now, we can find the surface area.

\[
SA = 2\pi(8)^2 + (16\pi)(21) \\
= 128\pi + 336\pi \\
= 464\pi \text{ units}^2
\]

**Example C**

Find the volume of the cylinder.

If the diameter is 16, then the radius is 8.

\[
V = \pi 8^2(21) = 1344\pi \text{ units}^3
\]
Cylinders CK-12

Guided Practice

1. Find the volume of the cylinder.

![Cylinder diagram]

2. If the volume of a cylinder is $484\pi \text{ in}^3$ and the height is 4 in, what is the radius?

3. The circumference of the base of a cylinder is $80\pi \text{ cm}$ and the height is 36 cm. Find the total surface area.

Answers:

1. $V = \pi r^2 (15) = 540\pi \text{ units}^3$

2. Solve for $r$.

\[
484\pi = \pi r^2 (4) \\
121 = r^2 \\
11\text{ in} = r
\]

3. We need to solve for the radius, using the circumference.

\[
2\pi r = 80\pi \\
r = 40
\]

Now, we can find the surface area.

\[
SA = 2\pi(40)^2 + (80\pi)(36) \\
= 3200\pi + 2880\pi \\
= 6080\pi \text{ units}^2
\]

Practice

1. Two cylinders have the same surface area. Do they have the same volume? How do you know?
2. A cylinder has \( r = h \) and the radius is 4 cm. What is the volume?

3. A cylinder has a volume of \( 486\pi \text{ ft}^3 \). If the height is 6 ft., what is the diameter?

4. A right cylinder has a 7 cm radius and a height of 18 cm. Find the volume.

Find the volume of the following solids. Round your answers to the nearest hundredth.

5. 

6. 

Find the value of \( x \), given the volume.

7. \( V = 6144\pi \text{ units}^3 \)

8. The area of the base of a cylinder is \( 49\pi \text{ in}^2 \) and the height is 6 in. Find the volume.

9. The circumference of the base of a cylinder is \( 34\pi \text{ cm} \) and the height is 20 cm. Find the total surface area.

10. The lateral surface area of a cylinder is \( 30\pi \text{ m}^2 \) and the height is 5m. What is the radius?
Here you’ll learn what a pyramid is and how to find its volume and surface area. What if you were given a solid three-dimensional figure with one base and lateral faces that meet at a common vertex? How could you determine how much two-dimensional and three-dimensional space that figure occupies? After completing this Concept, you’ll be able to find the surface area and volume of a pyramid.

**Watch This**

![Pyramids CK-12](multimedia)

**Guidance**

A **pyramid** is a solid with one **base** and **lateral faces** that meet at a common **vertex**. The edges between the lateral faces are **lateral edges**. The edges between the base and the lateral faces are **base edges**.

A **regular pyramid** is a pyramid where the base is a regular polygon. All regular pyramids also have a **slant height**, which is the height of a lateral face. A non-regular pyramid does not have a slant height.
Surface Area

Surface area is a two-dimensional measurement that is the total area of all surfaces that bound a solid. The basic unit of area is the square unit. For pyramids, we will need to use the slant height, which is labeled $l$, to find the area of each triangular face.

Surface Area of a Regular Pyramid: If $B$ is the area of the base, and $n$ is the number of triangles, then $SA = B + \frac{1}{2}nbl$.

The net shows the surface area of a pyramid. If you ever forget the formula, use the net.

Volume

To find the volume of any solid you must figure out how much space it occupies. The basic unit of volume is the cubic unit.

Volume of a Pyramid: $V = \frac{1}{3}Bh$ where $B$ is the area of the base.

Example A

Find the slant height of the square pyramid.

The slant height is the hypotenuse of the right triangle formed by the height and half the base length. Use the Pythagorean Theorem.
Example B

Find the surface area of the pyramid from Example A.

The total surface area of the four triangular faces is $4 \left(\frac{1}{2}bl\right) = 2(16) \left(8 \sqrt{10}\right) = 256 \sqrt{10}$. To find the total surface area, we also need the area of the base, which is $16^2 = 256$. The total surface area is $256 \sqrt{10} + 256 \approx 1065.54 \text{ units}^2$.

Example C

Find the volume of the pyramid.

$V = \frac{1}{3}(12^2)12 = 576 \text{ units}^3$

Pyramids CK-12

Guided Practice

1. Find the surface area of the regular triangular pyramid.
1.5. Pyramids

2. If the lateral surface area of a regular square pyramid is $72 \text{ ft}^2$ and the base edge is equal to the slant height. What is the length of the base edge?

3. Find the height and then volume of the pyramid.

4. Find the volume of the pyramid with a right triangle as its base.

5. A rectangular pyramid has a base area of $56 \text{ cm}^2$ and a volume of $224 \text{ cm}^3$. What is the height of the pyramid?

Answers:

1. “Regular” tells us the base is an equilateral triangle. Let’s draw it and find its area.

$$B = \frac{1}{2} \cdot 8 \cdot 4 \sqrt{3} = 16 \sqrt{3}$$
The surface area is:

$$SA = 16 \sqrt{3} + \frac{1}{2} \cdot 3 \cdot 8 \cdot 18 = 16 \sqrt{3} + 216 \approx 243.71$$

2. In the formula for surface area, the lateral surface area is \( \frac{1}{2}nb\). We know that \( n = 4 \) and \( b = l \). Let’s solve for \( b \).

$$\frac{1}{2}nb = 72 \text{ ft}^2$$

$$\frac{1}{2}(4)b^2 = 72$$

$$2b^2 = 72$$

$$b^2 = 36$$

$$b = 6 \text{ feet}$$

3. In this example, we are given the slant height. Use the Pythagorean Theorem.

$$7^2 + h^2 = 25^2$$

$$h^2 = 576$$

$$h = 24$$

$$V = \frac{1}{3}(14^2)(24) = 1568 \text{ units}^3.$$

4. The base of the pyramid is a right triangle. The area of the base is \( \frac{1}{2}(14)(8) = 56 \text{ units}^2 \).

$$V = \frac{1}{3}(56)(17) \approx 317.33 \text{ units}^3$$

5. Use the formula for volume and plug in the information we were given. Then solve for the height.

$$V = \frac{1}{3}Bh$$

$$224 = \frac{1}{3} \cdot 56h$$

$$12 = h$$

**Practice**

Fill in the blanks about the diagram to the left.
1. $x$ is the ___________.
2. The slant height is ________.
3. $y$ is the ___________.
4. The height is ________.
5. The base is ________.
6. The base edge is ________.

For questions 7-8, sketch each of the following solids and answer the question. Your drawings should be to scale, but not one-to-one. Leave your answer in simplest radical form.

7. Draw a square pyramid with an edge length of 9 in and a 12 in height. Find the slant height.
8. Draw an equilateral triangle pyramid with an edge length of 6 cm and a height of 6 cm. What is the height of the base?

Find the slant height, $l$, of one lateral face in each pyramid. Round your answer to the nearest hundredth.

9. [Diagram with slant height labeled 15]
10. [Diagram with slant height labeled 53]

Find the surface area and volume of the regular pyramid. Round your answers to the nearest hundredth.

11. [Diagram with surface area labeled 61, volume labeled 22]
12. [Diagram with surface area labeled 15, volume labeled 16]
17. A regular tetrahedron has four equilateral triangles as its faces.
   a. Find the height of one of the faces if the edge length is 6 units.
   b. Find the area of one face.
   c. Find the total surface area of the regular tetrahedron.

18. If the surface area of a square pyramid is $40 \text{ ft}^2$ and the base edge is 4 ft, what is the slant height?
19. If the lateral area of a square pyramid is $800 \text{ in}^2$ and the slant height is 16 in, what is the length of the base edge?
20. If the lateral area of a regular triangle pyramid is $252 \text{ in}^2$ and the base edge is 8 in, what is the slant height?
21. The volume of a square pyramid is 72 square inches and the base edge is 4 inches. What is the height?
22. The volume of a triangle pyramid is $170 \text{ in}^3$ and the base area is $34 \text{ in}^2$. What is the height of the pyramid?
1.6 Cones

Here you’ll learn what a cone is and how to find its volume and surface area.

What if you were given a three-dimensional solid figure with a circular base and sides that taper up towards a vertex? How could you determine how much two-dimensional and three-dimensional space that figure occupies? After completing this Concept, you’ll be able to find the surface area and volume of a cone.

Watch This

Cones CK-12

Guidance

A cone is a solid with a circular base and sides that taper up towards a vertex. A cone is generated from rotating a right triangle, around one leg. A cone has a slant height.

Surface Area

Surface area is a two-dimensional measurement that is the total area of all surfaces that bound a solid. The basic unit of area is the square unit. For the surface area of a cone we need the sum of the area of the base and the area of the sides.

Surface Area of a Right Cone: $SA = \pi r^2 + \pi rl$. 
Area of the base: $\pi r^2$

Area of the sides: $\pi rl$

**Volume**

To find the **volume** of any solid you must figure out how much space it occupies. The basic unit of volume is the cubic unit.

**Volume of a Cone:** $V = \frac{1}{3} \pi r^2 h$.

---

**Example A**

What is the surface area of the cone?

First, we need to find the slant height. Use the Pythagorean Theorem.

\[
l^2 = 9^2 + 21^2 = 81 + 441 = \sqrt{522} \approx 22.85
\]

The total surface area, then, is $SA = \pi 9^2 + \pi (9)(22.85) \approx 900.54 \text{ units}^2$. 
Example B

Find the volume of the cone.

First, we need the height. Use the Pythagorean Theorem.

\[5^2 + h^2 = 15^2\]
\[h = \sqrt{200} = 10\sqrt{2}\]
\[V = \frac{1}{3}(5^2)(10\sqrt{2})\pi \approx 370.24 \text{ units}^3\]

Example C

Find the volume of the cone.

We can use the same volume formula. Find the radius.

\[V = \frac{1}{3}\pi(3^2)(6) = 18\pi \approx 56.55 \text{ units}^3\]
Guided Practice

1. The surface area of a cone is $36\pi$ and the radius is 4 units. What is the slant height?
2. The volume of a cone is $484\pi \text{ cm}^3$ and the height is 12 cm. What is the radius?
3. Find the surface area and volume of the right cone. Round your answers to 2 decimal places.

Answers:

1. Plug what you know into the formula for the surface area of a cone and solve for $l$.

$$36\pi = \pi r^2 + \pi rl$$

$$36 = 16 + 4l$$  

$20 = 4l$  

$5 = l$

2. Plug what you know to the volume formula.

$$484\pi = \frac{1}{3} \pi r^2 (12)$$

$$121 = r^2$$

$11 \text{ cm} = r$

3. First we need to find the radius. Use the Pythagorean Theorem.

$$r^2 + 40^2 = 41^2$$

$$r^2 = 81$$

$r = 9$

Now use the formulas to find surface area and volume. Use the $\pi$ button on your calculator to help approximate your answer at the end.

$$SA = \pi r^2 + \pi rl$$

$$SA = 81\pi + 369\pi$$

$$SA = 450\pi$$

$$SA = 1413.72$$
1.6. Cones

Now for volume:

\[ V = \frac{1}{3} \pi r^2 h \]

\[ V = \frac{1}{3} \pi (9^2)(40) \]

\[ V = 1080\pi \]

\[ V = 3392.92 \]

**Practice**

Use the cone to fill in the blanks.

1. \( v \) is the ___________
2. The height of the cone is ______.
3. \( x \) is a __________ and it is the ___________ of the cone.
4. \( w \) is the _____________ ____________.

Sketch the following solid and answer the question. Your drawing should be to scale, but not one-to-one. Leave your answer in simplest radical form.

5. Draw a right cone with a radius of 5 cm and a height of 15 cm. What is the slant height?

Find the slant height, \( l \), of one lateral face in the cone. Round your answer to the nearest hundredth.

6. Find the surface area and volume of the right cones. Round your answers to 2 decimal places.
9. If the lateral surface area of a cone is \(30\pi \text{ cm}^2\) and the radius is 5 cm, what is the slant height?
10. If the surface area of a cone is \(105\pi \text{ cm}^2\) and the slant height is 8 cm, what is the radius?
11. If the volume of a cone is \(30\pi \text{ cm}^3\) and the radius is 5 cm, what is the height?
12. If the volume of a cone is \(105\pi \text{ cm}^3\) and the height is 35 cm, what is the radius?
1.7 Spheres

Here you’ll learn what a sphere is and how to find its volume and surface area.

What if you were given a solid figure consisting of the set of all points, in three-dimensional space, that are equidistant from a point? How could you determine how much two-dimensional and three-dimensional space that figure occupies? After completing this Concept, you’ll be able to find the surface area and volume of a sphere.

Watch This

Spheres CK-12

Guidance

A sphere is the set of all points, in three-dimensional space, which are equidistant from a point. The radius has one endpoint on the sphere and the other endpoint at the center of that sphere. The diameter of a sphere must contain the center.

A great circle is the largest circular cross-section in a sphere. The circumference of a sphere is the circumference of a great circle. Every great circle divides a sphere into two congruent hemispheres.
Surface Area

Surface area is a two-dimensional measurement that is the total area of all surfaces that bound a solid. The basic unit of area is the square unit. The best way to understand the surface area of a sphere is to watch the link by Russell Knightley, http://www.rkm.com.au/ANIMATIONS/animation-Sphere-Surface-Area-Derivation.html

Surface Area of a Sphere: \(SA = 4\pi r^2\).

![Sphere](image)

Volume

To find the volume of any solid you must figure out how much space it occupies. The basic unit of volume is the cubic unit. To see an animation of the volume of a sphere, see http://www.rkm.com.au/ANIMATIONS/animation-Sphere-Volume-Derivation.html by Russell Knightley.

Volume of a Sphere: \(V = \frac{4}{3}\pi r^3\).

![Sphere](image)

Example A

The circumference of a sphere is \(26\pi\) feet. What is the radius of the sphere?
The circumference is referring to the circumference of a great circle. Use \(C = 2\pi r\).

\[
2\pi r = 26\pi \\
r = 13\text{ ft}.
\]

Example B

Find the surface area of a sphere with a radius of 14 feet.

Use the formula.

\[
SA = 4\pi(14)^2 \\
= 784\pi\text{ ft}^2
\]
Example C

Find the volume of a sphere with a radius of 6 m.

Use the formula for volume:

\[ V = \frac{4}{3} \pi r^3 \]

\[ = \frac{4}{3} \pi (216) \]

\[ = 288 \pi m^3 \]

Guided Practice

1. Find the surface area of the figure below, a hemisphere with a circular base added.

2. The surface area of a sphere is 100\( \pi \) \text{in}^2. What is the radius?

3. A sphere has a volume of 14,137.167 \text{ft}^3. What is the radius?

Answers:

1. Use the formula for surface area.

\[ SA = \pi r^2 + \frac{1}{2} 4\pi r^2 \]

\[ = \pi (6^2) + 2\pi (6^2) \]

\[ = 36\pi + 72\pi = 108\pi \text{cm}^2 \]

2. Use the formula for surface area.
\[ SA = 4\pi r^2 \]
\[ 100\pi = 4\pi r^2 \]
\[ 25 = r^2 \]
\[ 5 = r \]

3. Use the formula for volume, plug in the given volume and solve for the radius, \( r \).

\[ V = \frac{4}{3}\pi r^3 \]
\[ 14,137.167 = \frac{4}{3}\pi r^3 \]
\[ \frac{3}{4\pi} \cdot 14,137.167 = r^3 \]
\[ 3375 \approx r^3 \]

At this point, you will need to take the \textit{cubed root} of 3375. Your calculator might have a button that looks like \( \sqrt[3]{\ } \), or you can do \( 3375^{\frac{1}{3}} \).

\[ \sqrt[3]{3375} = 15 \approx r \]

\[ \]

**Practice**

1. Are there any cross-sections of a sphere that are not a circle? Explain your answer.
2. List all the parts of a sphere that are the \textit{same} as a circle.
3. List any parts of a sphere that a circle does not have.

Find the surface area and volume of a sphere with: (Leave your answer in terms of \( \pi \))

4. a radius of 8 in.
5. a diameter of 18 cm.
6. a radius of 20 ft.
7. a diameter of 4 m.
8. a radius of 15 ft.
9. a diameter of 32 in.
10. a circumference of 26\( \pi \) cm.
11. a circumference of 50\( \pi \) yds.
12. The surface area of a sphere is 121\( \pi \) in\(^2\). What is the radius?
13. The volume of a sphere is 47916\( \pi \) m\(^3\). What is the radius?
14. The surface area of a sphere is 4\( \pi \) ft\(^2\). What is the volume?
15. The volume of a sphere is 36\( \pi \) m\(^3\). What is the surface area?
16. Find the radius of the sphere that has a volume of 335 cm\(^3\). Round your answer to the nearest hundredth.
17. Find the radius of the sphere that has a surface area 225\( \pi \) ft\(^2\).

Find the surface area and volume of the following shape. Leave your answers in terms of \( \pi \).
18. \( 45 \text{ cm.} \)
Here you’ll learn what a composite solid is and how to find its volume and surface area.

What if you built a solid three-dimensional house model consisting of a pyramid on top of a square prism? How could you determine how much two-dimensional and three-dimensional space that model occupies? After completing this Concept, you’ll be able to find the surface area and volume of composite solids like this one.

**Watch This**

Composite Solids CK-12

**Guidance**

A **composite solid** is a solid that is composed, or made up of, two or more solids. The solids that it is made up of are generally prisms, pyramids, cones, cylinders, and spheres. In order to find the surface area and volume of a composite solid, you need to know how to find the surface area and volume of prisms, pyramids, cones, cylinders, and spheres. For more information on any of those specific solids, consult the concept that focuses on them. This concept will assume knowledge of those five solids.

Most composite solids problems that you will see will be about volume, so most of the examples and practice problems below are about volume. There is one surface area example as well.

**Example A**

Find the volume of the solid below.

This solid is a parallelogram-based prism with a cylinder cut out of the middle.

\[ V_{prism} = (25 \cdot 25)30 = 18,750 \text{ cm}^3 \]

\[ V_{cylinder} = \pi(4)^2(30) = 480\pi \text{ cm}^3 \]

The total volume is \(18750 - 480\pi \approx 17,242.04 \text{ cm}^3\).
Example B

Find the volume of the composite solid. All bases are squares.

This is a square prism with a square pyramid on top. First, we need the height of the pyramid portion. Using the Pythagorean Theorem, we have, \( h = \sqrt{25^2 - 24^2} = 7 \).

\[
V_{\text{prism}} = (48)(48)(18) = 41,472 \text{ cm}^3
\]
\[
V_{\text{pyramid}} = \frac{1}{3}(48^2)(7) = 5376 \text{ cm}^3
\]

The total volume is \( 41,472 + 5376 = 46,848 \text{ cm}^3 \).

Example C

Find the surface area of the following solid.

This solid is a cylinder with a hemisphere on top. It is one solid, so do not include the bottom of the hemisphere or the top of the cylinder.

\[
SA = LA_{\text{cylinder}} + LA_{\text{hemisphere}} + A_{\text{base circle}}
\]
\[
= 2\pi rh + \frac{1}{2}4\pi r^2 + \pi r^2
\]
\[
= 2\pi(6)(13) + 2\pi 6^2 + \pi 6^2
\]
\[
= 156\pi + 72\pi + 36\pi
\]
\[
= 264\pi \text{ in}^2
\]
Guided Practice

1. Find the volume of the following solid.

![Diagram of a cylinder with a hemisphere on top. Dimensions: base radius 6 inches, height 13 inches.]

2. Find the volume of the base prism. Round your answer to the nearest hundredth.

![Diagram of a prism with dimensions 4 inches (base), 6 inches (height), and 5 inches (height of the triangular faces).]

3. Using your work from #2, find the volume of the pyramid and then of the entire solid.

Answers:

1. Use what you know about cylinders and spheres. The top of the solid is a hemisphere.

\[ V_{cylinder} = \pi \cdot 6^2 \cdot 13 = 468\pi \]
\[ V_{hemisphere} = \frac{1}{2} \left( \frac{4}{3} \pi \cdot 6^3 \right) = 144\pi \]
\[ V_{total} = 468\pi + 144\pi = 612\pi \text{ in}^3 \]

2. Use what you know about prisms.

\[ V_{prism} = B \cdot h \]
\[ V_{prism} = (4 \cdot 4) \cdot 5 \]
\[ V_{prism} = 80\text{in}^3 \]
3. Use what you know about pyramids.

\[ V_{\text{pyramid}} = \frac{1}{3}B \cdot h \]

\[ V_{\text{pyramid}} = \frac{1}{3} (4 \cdot 4)(6) \]

\[ V_{\text{pyramid}} = 32 \text{in}^3 \]

Now find the total volume by finding the sum of the volumes of each solid.

\[ V_{\text{total}} = V_{\text{prism}} + V_{\text{pyramid}} \]

\[ V_{\text{total}} = 112 \text{in}^3 \]

**Practice**

Round your answers to the nearest hundredth. The solid below is a cube with a cone cut out.

1. Find the volume of the cube.
2. Find the volume of the cone.
3. Find the volume of the entire solid.

The solid below is a cylinder with a cone on top.

4. Find the volume of the cylinder.
5. Find the volume of the cone.
6. Find the volume of the entire solid.
9. You may assume the bottom is open.

Find the volume of the following shapes. Round your answers to the nearest hundredth.
13. A sphere has a radius of 5 cm. A right cylinder has the same radius and volume. Find the height of the cylinder.

The bases of the prism are squares and a cylinder is cut out of the center.

14. Find the volume of the prism.
15. Find the volume of the cylinder in the center.
16. Find the volume of the figure.

This is a prism with half a cylinder on the top.

17. Find the volume of the prism.
18. Find the volume of the half-cylinder.
19. Find the volume of the entire figure.

Tennis balls with a 3 inch diameter are sold in cans of three. The can is a cylinder. Round your answers to the nearest hundredth.
20. What is the volume of one tennis ball?
21. What is the volume of the cylinder?
22. Assume the balls touch the can on the sides, top and bottom. What is the volume of the space not occupied by the tennis balls?
1.9 Area and Volume of Similar Solids

Here you’ll learn that the ratio of the surface areas of similar solids is equal to the square of their scale factor and that the ratio of their volumes is equal to the cube of their scale factor.

What if you were given two similar square prisms and told what the scale factor of their sides was? How could you find the ratio of their surface areas and the ratio of their volumes? After completing this Concept, you’ll be able to use the Surface Area Ratio and the Volume Ratio to solve problems like this.

Watch This

Similar Solids CK-12

Guidance

Two shapes are similar if all their corresponding angles are congruent and all their corresponding sides are proportional. **Two solids are similar** if they are the same type of solid and their corresponding radii, heights, base lengths, widths, etc. are proportional.

Surface Areas of Similar Solids

In two dimensions, when two shapes are similar, the ratio of their areas is the square of the scale factor. This relationship holds in three dimensions as well.

**Surface Area Ratio:** If two solids are similar with a scale factor of \( \frac{a}{b} \), then the surface areas are in a ratio of \( \left( \frac{a}{b} \right)^2 \).

Volumes of Similar Solids

Just like surface area, volumes of similar solids have a relationship that is related to the scale factor.

**Volume Ratio:** If two solids are similar with a scale factor of \( \frac{a}{b} \), then the volumes are in a ratio of \( \left( \frac{a}{b} \right)^3 \).

Summary

**Table 1.2:**

<table>
<thead>
<tr>
<th>Scale Factor</th>
<th>Ratios</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \frac{a}{b} )</td>
<td>( in, ft, cm, m, \text{ etc.} )</td>
</tr>
<tr>
<td>Ratio of the Surface Areas</td>
<td>( \left( \frac{a}{b} \right)^2 )</td>
<td>( in^2, ft^2, cm^2, m^2, \text{ etc.} )</td>
</tr>
<tr>
<td>Ratio of the Volumes</td>
<td>( \left( \frac{a}{b} \right)^3 )</td>
<td>( in^3, ft^3, cm^3, m^3, \text{ etc.} )</td>
</tr>
</tbody>
</table>
Example A

Are the two rectangular prisms similar? How do you know?

Match up the corresponding heights, widths, and lengths.

\[
\text{small prism} : \frac{3}{4.5} = \frac{4}{6} = \frac{5}{7.5}
\]

The congruent ratios tell us the two prisms are similar.

Example B

Two similar cylinders are below. If the ratio of the areas is 16:25, what is the height of the taller cylinder?

First, we need to take the square root of the area ratio to find the scale factor, \(\sqrt{\frac{16}{25}} = \frac{4}{5}\). Set up a proportion to find \(h\).

\[
\frac{4}{5} = \frac{24}{h}
\]

\[
4h = 120
\]

\[h = 30 \text{ units}\]

Example C

Two spheres have radii in a ratio of 3:4. What is the ratio of their volumes?

If we cube 3 and 4, we will have the ratio of the volumes. \(3^3 : 4^3 = 27 : 64\).
1.9. Area and Volume of Similar Solids

Guided Practice

1. Determine if the two triangular pyramids are similar.

![Triangular Pyramids](image)

2. Using the cylinders from Example B, if the area of the smaller cylinder is $1536\pi \text{ cm}^2$, what is the area of the larger cylinder?

3. If the ratio of the volumes of two similar prisms is 125:8, what is the scale factor?

4. Two similar triangular prisms are below. If the ratio of the volumes is 343:125, find the missing sides in both triangles.

![Triangular Prisms](image)

**Answers:**

1. Match up the corresponding parts.

   \[
   \frac{6}{8} = \frac{3}{4} = \frac{12}{16} \text{ however, } \frac{8}{12} = \frac{2}{3}.
   \]

   These triangle pyramids are not similar.

2. Set up a proportion using the ratio of the areas, 16:25.

   \[
   \frac{16}{25} = \frac{1536\pi}{A}
   \]

   \[
   16A = 38,400\pi
   \]

   \[
   A = 2400\pi \text{ cm}^2
   \]

3. Take the **cube root** of 125 and 8 to find the scale factor.
\[ \sqrt[3]{125} : \sqrt[3]{8} = 5 : 2 \]

4. The scale factor is 7:5, the cube root of 343:125. With the scale factor, we can now set up several proportions.

\[
\begin{align*}
\frac{7}{5} &= \frac{7}{y} & \frac{7}{5} &= \frac{x}{10} & \frac{7}{5} &= \frac{35}{w} & 7^2 + x^2 &= z^2 & \frac{7}{5} &= \frac{z}{v} \\
y &= 5 & x &= 14 & w &= 25 & 7^2 + 14^2 &= z^2 & \frac{7}{5} &= \frac{7 \sqrt{5}}{v} \rightarrow v = 5 \sqrt{5} \\
& & & & & z = \sqrt{245} = 7 \sqrt{5} \end{align*}
\]

**Practice**

Determine if each pair of right solids are similar.

5. Are all cubes similar? Why or why not?
6. Two prisms have a scale factor of 1:4. What is the ratio of their surface areas?
7. Two pyramids have a scale factor of 2:7. What is the ratio of their volumes?
8. Two spheres have radii of 5 and 9. What is the ratio of their volumes?
9. The surface area of two similar cones is in a ratio of 64:121. What is the scale factor?
10. The volume of two hemispheres is in a ratio of 125:1728. What is the scale factor?
11. A cone has a volume of $15\pi$ and is similar to another larger cone. If the scale factor is 5:9, what is the volume of the larger cone?

12. The ratio of the volumes of two similar pyramids is 8:27. What is the ratio of their total surface areas?

13. The ratio of the volumes of two tetrahedrons is 1000:1. The smaller tetrahedron has a side of length 6 cm. What is the side length of the larger tetrahedron?

14. The ratio of the surface areas of two cubes is 64:225. What is the ratio of the volumes?

Below are two similar square pyramids with a volume ratio of 8:27. The base lengths are equal to the heights. Use this to answer questions 15-18.

15. What is the scale factor?

16. What is the ratio of the surface areas?

17. Find $h, x$ and $y$.

18. Find the volume of both pyramids.

Use the hemispheres below to answer questions 19-20.

19. Are the two hemispheres similar? How do you know?

20. Find the ratio of the surface areas and volumes.

21. The ratio of the surface areas of two similar cylinders is 16:81. What is the ratio of the volumes?

Summary

This chapter presents three-dimensional geometric figures beginning with polyhedrons, regular polyhedrons, and an explanation of Euler’s Theorem. Three-dimensional figures represented as cross sections and nets are discussed. Then the chapter branches out to the formulas for surface area and volume of prisms, cylinders, pyramids, cones, spheres and composite solids. The relationship between similar solids and their surface areas and volumes are explored.