Section 7.1: Conic Basics

**Midpoint:** midpoint of a line segment is \((x, y) = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)\)

**Distance:** \(d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\)

**Conics:** the intersection of a plane and a cone. Note: conic is derived from the word cone.

![Conic Sections Diagram]

**General form of a 2nd degree equation in two variables:**
\[Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0\]

The **discriminant** \((B^2 - 4AC)\) determines the shape of a conic section:

<table>
<thead>
<tr>
<th>Conic</th>
<th>Discriminant</th>
<th>Geometric Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ellipse</td>
<td>(B^2 - 4AC &lt; 0)</td>
<td>the sum of whose distances to two fixed points is constant</td>
</tr>
<tr>
<td>Parabola</td>
<td>(B^2 - 4AC = 0)</td>
<td>equidistant to both a line and a fixed point</td>
</tr>
<tr>
<td>Hyperbola</td>
<td>(B^2 - 4AC &gt; 0)</td>
<td>the difference of whose distances to two fixed points is constant</td>
</tr>
</tbody>
</table>

**Note:** all circles are ellipses, so \(B^2 - 4AC < 0\) for them also.
Section 7.2: The Parabola

Definition of a Parabola: given a fixed point \( f \) (called the focus) and fixed line \( D \) (called the directrix) in the plane, then a parabola is the set of all points \((x, y)\) such that the distance from the focus, \( p \), to a point on the graph, \((x, y)\), is equal to the distance from the directrix to \((x, y)\).

**Vertical parabolas:** \( y = ax^2 + bx + c \)
1. Opens upward if \( a > 0 \) and downward if \( a < 0 \).
2. y-intercept at \((0, c)\)
3. x-intercept, set \( y = 0 \) and solve for \( x \)
4. Axis of symmetry: \( x = -\frac{b}{2a} \)
5. Vertex: \( \left(-\frac{b}{2a}, f \left(-\frac{b}{2a}\right)\right)\)
6. If the vertex is at \((h, k)\), then the vertex form is \( a(x - h)^2 + k = y \)
7. Using the directrix, can be rewritten as \( (x - h)^2 = 4p(y - k) \), \(|p|\) is the vertical distance from the focus to the vertex and from the vertex to the directrix.
8. If \( p > 0 \), then the parabola opens up and the focus is above the vertex \( p \) units and if \( p < 0 \), then the parabola opens downward and the focus is \(|p|\) units below the vertex.
9. There are two points on the graph which are each \(|2p|\) units from the focus, left and right.

**Horizontal parabolas:** \( x = ay^2 + by + c \)
1. Opens to the right if \( a > 0 \) and opens to the left if \( a < 0 \).
2. x-intercept at \((c, 0)\)
3. y-intercept, set \( x = 0 \) and solve for \( y \)
4. Axis of symmetry: \( y = -\frac{b}{2a} \)
5. Vertex: \( f \left(-\frac{b}{2a}, -\frac{b}{2a}\right)\)
6. If the vertex is at \((h, k)\), then the vertex form is \( a(y - k)^2 + h = x \)
7. Using the directrix, can be rewritten as \( (y - k)^2 = 4p(x - h) \), \(|p|\) is the horizontal distance from the focus to the vertex and from the vertex to the directrix.
8. If \( p > 0 \), then the parabola opens to the right and the focus is to the right of the vertex \( p \) units and if \( p < 0 \), then the parabola opens to left and the focus is \(|p|\) units to the left of the vertex.
9. There are two points on the graph which are each \(|2p|\) units from the focus, up and down.

**Note:** converting from vertex form to standard (directrix) form for the vertical parabola:
\[ a(x - h)^2 + k = y \]
\[ a(x - h)^2 = (y - k) \]
\[ (x - h)^2 = \frac{1}{a} (y - k) \Rightarrow 4p = \frac{1}{a} \rightarrow p = \frac{1}{4a} \]
Example: Find the vertex, focus and directrix for the parabola whose equation is $x^2 - 6x + 12y - 15 = 0$

Solution:

→ isolate the x and y terms and then complete the square for the x terms
→ $x^2 - 6x = -12y + 15$
→ $x^2 - 6x + 9 = -12y + 15 + 9$
→ $(x - 3)^2 = -12y + 24$
→ $(x - 3)^2 = -12(y - 2)$
→ Vertex: (3, 2)
→ Opens down
→ $4p = -12$ → $p = -3$ → focus is at $(3, 2 - 3) = (3, -1)$
→ Directrix will be at $y = 2 + 3$ → $y = 5$
→ Horizontal distance from the focus to the graph is $2|p| = 6$ → additional points on the graph are (9, −1) and (−3, −1).

Note: the Latus Rectum is the line segment through the focus of a parabola perpendicular to the major axis with endpoints on the parabola. The length of the latus rectum = $4|p|$

Example: Find the vertex, focus and directrix for the parabola whose equation is $y^2 - 12y - 20x + 36 = 0$
Example: Find the vertex, focus and directrix for the parabola whose equation is \( y^2 - 12y - 20x + 36 = 0 \).

Solution:
→ Isolate the x and y terms and complete the square for the y terms.
→ \( y^2 - 12y - 20x + 36 = 0 \)
→ \( y^2 - 12y = 20x - 36 \)
→ \( y^2 - 12y + 36 = 20x - 36 + 36 \)
→ \( (y - 6)^2 = 20x \)
→ Vertex: \((0, 6)\)
→ Opens to the right
→ \( 4p = 20 \) → \( p = 5 \) → focus is at \((0 + 5, 6) = (5, 6)\)
→ Directrix will be at \( x = 0 - 5 \) → \( x = -5 \)
→ Vertical distance from the focus to the graph is \(|2p| = 10\)
→ Additional points on the graph are at \((5, 16)\) and \((5, -4)\)

Parabolic Receivers – if incoming beams of light or radio waves are parallel to the axis of symmetry of a parabolic receiver, they will be reflected toward the focus as indicated below.

Example: Find the distance the focus should be placed from the vertex (base of the receiver), if one point on the receiver is located 2 meters above and 7 meters to the right of the vertex.

Solution: use the following equation: \( x^2 = 4py \) → \( 7^2 = 4p(2) \) → \( p = \frac{49}{8} \) → \( p = 6.125 \) meters → the focus should be placed 6.125 meters directly above the vertex.
Section 7.3: The Ellipse

Definition of a Circle: is the set of all points that are an equal distance (called the radius) from a given point (called the center).

Equation for a circle: \((x - h)^2 + (y - k)^2 = r^2\); center at \((h, k)\) and radius = \(r\).

Example: fine the equation of a circle with a center at \((3, -2)\) and radius = 5
Solution: \((x - 3)^2 + (y - (-2))^2 = 5^2 \rightarrow (x - 3)^2 + (y + 2)^2 = 25\)

Standard form of a circle: divide both sides by \(r^2\) \(\frac{(x-h)^2}{r^2} + \frac{(y-k)^2}{r^2} = 1\)
Example: find the center and radius of a circle by completing the square: \(x^2 + y^2 - 6x + 4y - 3 = 0\) and write the equation in standard form.

Example: find the center and radius of a circle by completing the square: \(x^2 + y^2 - 6x + 4y - 3 = 0\)
Solution:
→ push the constant to the right-hand side and group the x terms and y terms: \(x^2 - 6x + y^2 + 4y = 3\)
→ complete the square, make sure the coefficient of the squared term is 1
→ then add \((b/2)^2\) to each side of the equation.
→ \((x^2 - 6x + (6/2)^2) + (y^2 + 4y + (4/2)^2) = 3 + (6/2)^2 + (4/2)^2\)
→ \((x^2 - 6x + 9) + (y^2 + 4y + 4) = 3 + 9 + 4\) and now you have 2 sets of perfect squares.
→ \((x - 3)^2 + (y + 2)^2 = 3 + 9 + 4 \rightarrow (x - 3)^2 + (y + 2)^2 = 16\)
→ center is at \((3, -2)\) and the radius = 4
→ \(\frac{(x-3)^2}{4^2} + \frac{(y+2)^2}{4^2} = 1\)
**Definition of an Ellipse:** Given two points, $f_1$ and $f_2$, each called a focus (plural is foci) in a plane. Then an ellipse is the set of all points $(x, y)$ where the distance from $f_1$ to $(x, y)$ added to the distance from $f_2$ to $(x, y)$ remains constant.

![Diagram of an ellipse with foci, vertices, and center labeled]

**Standard form of an Ellipse:**

\[
\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1
\]

1. $(h, k)$ is the center of the ellipse.
2. $|a|$ is the horizontal distance from the center to the graph, left edge $(h - |a|, k)$; right edge $(h + |a|, k)$
3. $|b|$ is the vertical distance from the center to the graph, top edge $(h, k + |b|)$; bottom edge $(h, k - |b|)$
4. Foci: there are two fixed points $f_1$ and $f_2$, each called a focus, such that an ellipse is the set of all points $(x, y)$ where the sum of the distance from $f_1$ to $(x, y)$ and the distance from $f_2$ to $(x, y)$ is constant.
5. If $a > b$, then the major axis is horizontal, parallel to the x-axis. And $c^2 = |a^2 - b^2| \rightarrow$ foci are at $(h - c, k)$ and at $(h + c, k)$
6. If $b > a$, then the major axis is vertical, parallel to the y-axis. And $c^2 = |a^2 - b^2| \rightarrow$ foci are at $(h, k + c)$ and at $(h, k - c)$

**Example:** find the equation of an ellipse and locate the foci, given:

\[25x^2 + 9y^2 - 100x - 54y - 44 = 0\]
Example: find the equation of an ellipse and locate the foci, given \(25x^2 + 9y^2 − 100x − 54y − 44 = 0\)

Solution:

→ factor and group: \(25(x^2 − 4x) + 9(y^2 − 6y) = 44\)

→ complete the square: \(25(x^2 − 4x + 4) + 9(y^2 − 6y + 9) = 44 + 25(4) + 9(9)\)

→\(25(x−2)^2 + 9(y−3)^2 = 44 + 100 + 81\)

→ divide both sides by 225: \(\frac{25(x−2)^2}{225} + \frac{9(y−3)^2}{225} = 1\)

→\(\frac{(x−2)^2}{9} + \frac{(y−3)^2}{25} = 1\)

→ center is at (2, 3)

→ and since \(b > a\) → major axis is vertical, parallel to the y-axis.

→ also, \(c^2 = |a^2 − b^2|\) → \(3^2 − 5^2\) → \(c = 4\)

→ foci are at (2, 3 + 4) and at (2, 3 − 4)

→ (2, 7) and (2, −1)

Example: Find the equation of an ellipse (in standard form) that has foci at (3, 2) and (3, 6) with a minor axis 6 units in length.

Solution:

→ the major axis will be vertical, and the center of the ellipse will be the midpoint of the two foci

→ (3, 4) → \(c = 2\)

→ Also, since the minor axis is 6 units in length

→ \(2a = 6\) → \(a = 3\)

→ In general: \(c^2 = |a^2 − b^2|\) → \(2^2 = |3^2 − 5^2|\) → \(4 = |9 − b^2|\)

→ we know that \(b > a\) → \(−4 = 9 − b^2\) → \(b^2 = 13\)

→ \(b = \sqrt{13}\)

→ \(\frac{(x−3)^2}{3^2} + \frac{(y−4)^2}{(\sqrt{13})^2} = 1\)
Section 7.4: The Hyperbola

Definition of a Hyperbola: Given two points, \( f_1 \) and \( f_2 \), in a plane, a hyperbola is the set of all points \((x, y)\) where the distance from \( f_2 \) to \((x, y)\) subtracted from the distance from \( f_1 \) to \((x, y)\) is a positive constant. The points, \( f_1 \) and \( f_2 \), are called the foci of the hyperbola.

![Diagram of a hyperbola with foci \( f_1 \) and \( f_2 \), a point \( P \), and distances \( d_1 \) and \( d_2 \).]

Standard forms of a Hyperbola: \[
\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \text{ or } \frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1
\]

1st case: \[
\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1
\]

1. Horizontal hyperbola with a center at \((h, k)\), symmetric about a horizontal line (called the Transverse axis).
2. Transverse axis is \( y = k \) (line passes through the center and both vertices).
3. Conjugate axis is \( x = h \) (line passes through the center and is perpendicular to the Transverse axis).
4. \( |a| \) gives the distance from the center to the vertices
5. Asymptotes can be drawn starting at \((h, k)\) and using slopes, \( m \), where \( m = \pm\frac{b}{a} \)
6. Foci: there are two fixed points \( f_1 \) and \( f_2 \), each called a focus, such that a hyperbola is the set of all points \((x, y)\) such that the distance from \( f_2 \) to \((x, y)\) subtracted from the distance from \( f_1 \) to \((x, y)\) is a positive constant. \( c \) is the distance from the center to each focus and \( c^2 = a^2 + b^2 \)
2nd case: \( \frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1 \)

1. Vertical hyperbola with a center at \((h, k)\), symmetric about a vertical line (called the Transverse axis).
2. Conjugate axis is \(y = k\) (line passes through the center and is perpendicular to the Transverse axis).
3. Transverse axis is \(x = h\) (line passes through the center and both vertices).
4. \(|b|\) gives the distance from the center to the vertices.
5. Asymptotes can be drawn starting at \((h, k)\) and using slopes, \(m\), where \(m = \pm \frac{b}{a}\).

**Example:** graph the hyperbola represented by 

\[ 16(x - 2)^2 - 9(y - 1)^2 = 144 \]
Example: graph the hyperbola represented by \(16(x - 2)^2 - 9(y - 1)^2 = 144\)

Solution: Divide both sides by 144 → \(\frac{(x-2)^2}{3^2} - \frac{(y-1)^2}{4^2} = 1\)

→ Center: (2, 1)

→ \(a = 3\)

→ vertices are at (−1, 1) and (5, 1)

→ Asymptotes have a slope of \(m = \pm \frac{4}{3}\)

Example: graph the equation \(9y^2 - x^2 + 54y + 4x + 68 = 0\)

Solution:

\[
\begin{align*}
\rightarrow & \quad 9y^2 + 54y - x^2 + 4x = -68 \\
\rightarrow & \quad 9(y^2 + 6y) - (x^2 - 4x) = -68 \\
\rightarrow & \quad 9(y^2 + 6y + 9) - (x^2 - 4x + 4) = -68 + 81 - 4 \\
\rightarrow & \quad 9(y + 3)^2 - (x - 2)^2 = 9 \\
\rightarrow & \quad \frac{(y+3)^2}{3^2} - \frac{(x-2)^2}{3^2} = 1 \\
\rightarrow & \quad Center is at (2, -3) \\
\rightarrow & \quad a = 3, b = 1 \\
\rightarrow & \quad c^2 = a^2 + b^2 \rightarrow c^2 = 9 + 1 \rightarrow c = \sqrt{10} \\
\rightarrow & \quad Slope of the asymptotes: m = \pm \frac{b}{a} \rightarrow m = \pm \frac{1}{3}
\end{align*}
\]
Section 7.5: Systems of Nonlinear Equations

Example: Solve the system of equations using substitution:
\[ y = x^2 - 2x - 3 \]
\[ 2x - y = 7 \]

Solution:
→ 2nd equation: \( y = 2x - 7 \), substitute that into the 1st equation for \( y \)
→ \( 2x - 7 = x^2 - 2x - 3 \)
→ \( x^2 - 4x + 4 = 0 \)
→ \( (x - 2)(x - 2) = 0 \)
→ \( x = 2 \) is the solution for \( x \), find \( y \):
→ \( 2x - y = 7 \) → \( 2(2) - y = 7 \) → \( y = -3 \)
→ \( (2, -3) \) is the solution

Example: Solve the system of equations using elimination:
\[ 2y^2 - 5x^2 = 13 \]
\[ 3x^2 + 4y^2 = 39 \]

Solution:
→ \(-2(-5x^2 + 2y^2 = 13) \rightarrow 10x^2 - 4y^2 = -26 \)
→ \( 3x^2 + 4y^2 = 39 \) → \( 3x^2 + 4y^2 = 39 \) (add the 2 equations)
→ \( 13x^2 = 13 \rightarrow x^2 = 1 \rightarrow x = \pm1 \)
→ Solve for \( y \):
→ \( 3 + 4y^2 = 39 \)
→ \( 4y^2 = 36 \)
→ \( y^2 = 9 \)
→ \( y = \pm3 \)
→ the solutions are: \((-1, -3); (-1, 3); (1, -3)\) and \((1, 3)\)

Example: Solve the system of logarithmic equations:
\[ y = -\log(x + 7) + 2 \]
\[ y = \log(x + 4) + 1 \]
Example: Solve the system of logarithmic equations:
\[ y = -\log(x + 7) + 2 \]
\[ y = \log(x + 4) + 1 \]

Solution:
→ Set the two equations equal to one another
→ \(-\log(x + 7) + 2 = \log(x + 4) + 1\)
→ \(-\log(x + 7) - \log(x + 4) = -1\)
→ \(\log(x + 7) + \log(x + 4) = 1\)
→ \((x + 7)(x + 4) = 10^1\)
→ \(x^2 + 11x + 28 = 10\)
→ \(x^2 + 11x + 18 = 0\)
→ \((x + 2)(x + 9) = 0\)
→ \(x = -2\) or \(x = -9\)
→ \(-9\) cannot be a solution because you would be taking the log of a negative number
→ therefore, the only solution is \(x = -2\)
→ \(y = \log(x + 4) + 1 \rightarrow y = \log(-2 + 4) + 1 \rightarrow y = \log(2) + 1 = 1.301\)
→ the solution is \((-2, 1.301)\)

Section 7.6: System of Nonlinear Inequalities

Example: Find the solution region for the following system of inequalities:
\[ x^2 + 4y^2 < 25 \]
\[-x + 4y \geq 5 \]

Solution: first, determine where they intersect
→ \(x^2 + 4y^2 = 25\)
→ \(-x + 4y = 5 \rightarrow x = 4y - 5\), substitute for \(x\) back into the 1st equation and solve for \(y\)
→ \((4y - 5)^2 + 4y^2 = 25\)
→ \(16y^2 - 40y + 25 + 4y^2 = 25\)
→ \(20y^2 - 40y = 0\)
→ \(y^2 - 2y = 0\)
→ \(y(y - 2) = 0\)
→ \(y = 0\) or \(y = 2\); solve for \(x \rightarrow -x + 4y = 5 \rightarrow y = 0\), then \(x = -5\) and if \(y = 2\), then \(x = 3\)
→ Intersections are at \((-5, 0)\) and \((3, 2)\)
The solution is the sector above the purple line (including the purple line) and below the top portion of the ellipse, not including the ellipse, i.e., the shaded area.