Suppose \( f \) is a smooth, non-negative function on \([a,b]\) and that a surface of revolution is generated by revolving that portion of the curve \( y = f(x) \) between \( x = a \) and \( x = b \) about the \( x \)-axis.

We need to define what is meant by the areas of the surface and determine a formula for it.

Let's recall a cylinder. It was created by connecting the 2 sides of a rectangle.

Now we need to extend that concept to other shapes.
Consider this

\[ y \]
\[ a = x_0 \]
\[ b = x_n \]

Similar to shell method, but are combining exterior shells

So 

\[ S = \omega \int \text{arc length } ds \]

(see extra material for 8.1)

General Definition

Rotation about the \( x \)-axis between \( x = a \) and \( x = b \)

\[ S = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} \, dx \]

\( f \) is a smooth, non-negative function on \([a, b]\)

This definition can be customized into a more friendly format and also adjusted for rotation about the \( y \)-axis.
Rotation about the x-axis

\[ S = \int_{a}^{b} \pi f(x) \sqrt{1 + (f'(x))^2} \, dx \]

\[ S = \int_{a}^{b} \pi y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \]

It's important to note that the radius must be \( y \) when rotating about the x-axis.

Rotation about the y-axis

\[ S = \int_{c}^{d} \pi g(y) \sqrt{1 + (g'(y))^2} \, dy \]

\[ S = \int_{c}^{d} \pi x \sqrt{1 + \left( \frac{dx}{dy} \right)^2} \, dy \]

This part can be adjusted for ease of integration.

It's important to note that the radius must be \( x \) when rotating about the y-axis, but the relationship \( x = g(y) \) needs to be used instead.
Example 1: Find the area of the surface that is generated by revolving the portion of the curve $y = x^3$ between $x = 0$ and $x = 1$ about the x-axis.

$$S = 2\pi \int_a^b y \sqrt{1 + (\frac{dy}{dx})^2} \, dx$$

$$y = x^3 \quad \frac{dy}{dx} = 3x^2$$

The entire integral must be written in terms of $x$.

$$S = \int_0^1 2\pi (x^3) \sqrt{1 + (3x^2)^2} \, dx$$

$$S = \int_0^1 2\pi x^3 \sqrt{1 + 9x^4} \, dx$$

$$S = \frac{2\pi}{36} \left[ (1 + 9x^4)^{3/2} \right]_0^1$$

$$S = \frac{\pi}{27} \left[ (1 + 9(1)^4)^{3/2} - (1 + 9(0)^4)^{3/2} \right]$$

$$S = \frac{\pi}{27} \left[ 10^{3/2} - 1^{3/2} \right] = \frac{\pi}{27} (10^{3/2} - 1)$$

$$\approx 3.56$$
Example 1 - Alternate approach

\[ S = \int_a^b \pi y \sqrt{1 + (\frac{dx}{dy})^2} \, dy \]

\[ y = x^3 \]
\[ x = \sqrt[3]{y} = y^{\frac{1}{3}} \]

when \( x = 0 \), \( y = (0)^3 = 0 \)
when \( x = 1 \), \( y = (1)^3 = 1 \)

\[ \frac{dx}{dy} = \frac{1}{3} y^{-\frac{2}{3}} \]

\[ S = \pi \int_0^1 y \sqrt{1 + \left( \frac{1}{3} y^{-\frac{2}{3}} \right)^2} \, dy = \pi \int_0^1 y \sqrt{1 + \frac{1}{9y^{\frac{2}{3}}}} \, dy \]

\[ S = \pi \int_0^1 y \sqrt{y^{\frac{4}{3}} + 1} \, dy = \pi \int_0^1 \frac{u^{\frac{1}{3}}}{3y^{\frac{1}{3}}} (9y^{\frac{4}{3}} + 1)^{\frac{1}{2}} \, du \]

\[ S = \pi \int_0^1 y^{\frac{1}{3}} (9y^{\frac{4}{3}} + 1)^{\frac{1}{2}} \, dy \]

\[ u = 9y^{\frac{4}{3}} + 1 \]
\[ du = 12y^{\frac{1}{3}} \, dy \]
\[ \frac{du}{12} = y^{\frac{1}{3}} dy \]
\[ S = \frac{\pi}{18} \left[ 2 \left( 9y^{\frac{4}{3}} + 1 \right)^{\frac{3}{2}} \right]_0^1 \]

\[ S = \frac{\pi}{18} \left[ \left( 9(1)^{\frac{4}{3}} + 1 \right)^{\frac{3}{2}} - \left( 9(0)^{\frac{4}{3}} + 1 \right)^{\frac{3}{2}} \right] \]

\[ S = \frac{\pi}{18} \left[ (9+1)^{\frac{3}{2}} - (1)^{\frac{3}{2}} \right] \]

\[ S = \frac{\pi}{18} \left( 10^{\frac{3}{2}} - 1 \right) \approx 3.56 \]
Example 2

Find the area of the surface that is generated by revolving the portion of the curve \( y = x^2 \) between \( x = 1 \) and \( x = 2 \) about the y-axis.

Option 1

\[
S = \pi \int_{c}^{d} x \sqrt{1 + \left( \frac{dx}{dy} \right)^2} \, dy
\]

from \((1,1)\) to \((2,4)\)

\[
S = \pi \int_{1}^{4} (y^{1/2}) \sqrt{1 + \left( \frac{1}{2y^{1/2}} \right)^2} \, dy
\]

so since \( y = x^2 \) \( \frac{dx}{dy} = \frac{1}{2}y^{-1/2} \)

\[
S = \pi \int_{1}^{4} y^{1/2} \sqrt{1 + \frac{1}{4y}} \, dy = \pi \int_{1}^{4} y^{1/2} \sqrt{\frac{4y + 1}{4y}} \, dy
\]

\[
= \pi \int_{1}^{4} \frac{y^{1/2}}{2y^{1/2}} \sqrt{4y + 1} \, dy = \pi \int_{1}^{4} \frac{1}{2} (4y + 1)^{1/2} \, dy
\]

\[
= \frac{\pi}{2} \left[ (4y + 1)^{3/2} \right]_{1}^{4} = \frac{\pi}{6} \left[ (4(4) + 1)^{3/2} - (4(1) + 1)^{3/2} \right] = \frac{\pi}{6} \left[ 17^{3/2} - 5^{3/2} \right]
\]
Example 2 — Option 2

\[ S = 2\pi \int_C x \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \]

\[ y = x^2 \]
\[ x = \sqrt{y} \]
\[ \frac{dy}{dx} = 2x \]

(1,1) to (2,4)

\[ S = 2\pi \int_1^2 x \left( 1 + (2x)^2 \right)^{1/2} \, dx \]

\[ S = 2\pi \int_1^2 x \left( 1 + 4x^2 \right)^{1/2} \, dx \]

\[ S = \frac{2\pi}{8} \int_1^2 (1 + 4x^2)^{1/2} (8x \, dx) \]

\[ S = \frac{\pi}{4} \int_1^2 (1 + 4x^2)^{1/2} \, 6x \, dx \]

\[ S = \frac{\pi}{4} \left[ \frac{3}{2} (1 + 4x^2)^{3/2} \right]_1^2 \]

\[ S = \frac{\pi}{6} \left[ (1 + 4(2)^2)^{3/2} - (1 + 4(1)^2)^{3/2} \right] \]

\[ = \frac{\pi}{6} \left[ 17^{3/2} - 5^{3/2} \right] \]
Example 3

Find the area of the surface that is generated by revolving about the x-axis on the interval \([1, 2]\)

for revolving about x-axis we start with one of the following

\[
S = \int_{a}^{b} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \\
S = \int_{a}^{b} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy
\]

Option 1

\[
y = \frac{x^3}{6} + \frac{1}{2x}
\]

\[
\frac{dy}{dx} = \frac{3x^2}{6} + \frac{1}{2x^2}
\]

\[
\frac{dy}{dx} = \frac{1}{2} x^2 - \frac{1}{2} x^{-2}
\]

\[
x = 1 \quad y = \frac{1}{6} + \frac{1}{2} = \frac{2 + 6}{12} = \frac{8}{12} = \frac{2}{3}
\]

\[
x = 2 \quad y = \frac{(2)^3}{6} - \frac{1}{2} \cdot \frac{1}{2} = \frac{8 - 1}{6} = \frac{16 - 3}{12} = \frac{13}{12}
\]

alternative

\[
y = 2x^4 + 6
\]

\[
12xy = 2x^4 + 6
\]

\[
12xy - 2x^4 = 6
\]

we need to solve for x which is not user-friendly, so \textbf{option 1 is our best choice based on this data.}

\((1, \frac{3}{4}) \to (2, \frac{13}{12})\)
\[ S = 2\pi \int_{1}^{2} y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \]

\[ S = 2\pi \int_{1}^{2} \left( \frac{x^3}{6} + \frac{1}{2x} \right) \sqrt{1 + \left( \frac{x^2}{2} - \frac{1}{2x^2} \right)^2} \, dx \]

\[ S = 2\pi \int_{1}^{2} \left( \frac{x^3}{6} + \frac{1}{2x} \right) \sqrt{1 + \left( \frac{x^4}{4} - 2 \left( \frac{x^2}{2x} \right) \frac{1}{2x^2} + \frac{1}{4x^4} \right)} \, dx \]

\[ S = 2\pi \int_{1}^{2} \left( \frac{x^3}{6} + \frac{1}{2x} \right) \sqrt{1 + \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}} \, dx \]

\[ S = 2\pi \int_{1}^{2} \left( \frac{x^3}{6} + \frac{1}{2x} \right) \sqrt{\frac{1}{2} + \frac{x^4}{4} + \frac{1}{4x^4}} \, dx \, \text{, } \text{lcd} = 4x^4 \]

\[ S = 2\pi \int_{1}^{2} \left( \frac{x^3}{6} - \frac{1}{2x} \right) \sqrt{\frac{2x^4 + x^8 + 1}{4x^4}} \, dx \]

\[ S = 2\pi \int_{1}^{2} \left( \frac{x^3}{6} - \frac{1}{2x} \right) \sqrt{\frac{(x^4 + 1)^2}{4x^4}} \, dx \]

\[ S = 2\pi \int_{1}^{2} \left( \frac{x^3}{6} - \frac{1}{2x} \right) \frac{(x^4 + 1)}{2x^2} \, dx \]

\[ S = 2\pi \int_{1}^{2} \left( \frac{x^3}{6} - \frac{1}{2x} \right) \left( \frac{x^2}{2} + \frac{1}{2x^2} \right) \, dx = \frac{4711}{10} \]
Find the area of the surfaces generated given specific conditions.

1. \( y = \frac{1}{3} x^3 \) on the interval \([0, 3]\) about the \(x\)-axis

2. \( y = \sqrt[3]{x} + 2 \) on the interval \([1, 8]\) about the \(y\)-axis

3. \( y = \sin x \) on the interval \([0, \pi]\) about the \(x\)-axis

4. \( y = e^{-x} \) from \(x = 0\) to \(x = 1\) about the \(x\)-axis

5. \( y = \ln x \) from \(x = 1\) to \(x = 2\) about the \(y\)-axis

Use the calculator to integrate this one.
Section 8.2
Areas of surface of Revolution

practice problem answers and helpful hints

1. \[ S = \pi \int_{0}^{3} \frac{1}{3}x^3 \sqrt{1 + x^4} \, dx = \frac{\pi}{9} (8\sqrt[3]{8a} - 1) \equiv 258.85 \]

2. \[ S = \pi \int_{1}^{8} x \sqrt{1 + \frac{1}{9x^{4/3}}} \, dx = \frac{\pi}{27} (145\sqrt[3]{45} - 10\sqrt{10}) \equiv 199.48 \]

3. 14.424

4. \[ S = \pi \int_{0}^{1} e^{-x} \sqrt{1 + (e^{-x})^2} \, dx = \pi \int_{0}^{1} e^{-x} \sqrt{1 + e^{-2x}} \, dx \]

   hint: use trig substitution

   final answer \[ \pi \left[ \sqrt{a} + \ln(1+\sqrt{a}) - e^{-1} \sqrt{e^{2a}+1} - \ln(e^{-1}+\sqrt{e^{2a}+1}) \right] \]

5. also use trig substitution \[ S = \pi \int_{0}^{a} \sqrt{x^2 + 1} \, dx \]

   square both sides (s²) to make simplification easier

   final answer \[ \pi \left[ \sqrt{a} + \ln(a + \sqrt{a}) \right] - \left[ \sqrt{a} + \ln(1+\sqrt{a}) \right] \]
Further Exploration

Let's make some more connections to the surface area $ds$

Alternate version

Let a smooth curve $C$ be given by $x = f(t)$ and $y = g(t)$ [parametric equations] where $a \leq t \leq b$ and suppose that $g(t) \geq 0$ for all $t$. The area $S$ of the surface of revolution obtained by revolving curve $C$ about the $x$-axis is

$$S = \frac{\pi}{a} \sqrt{(f'(t))^2 + (g'(t))^2} \, dt = \frac{\pi}{a} \int_{a}^{b} \sqrt{(dx)^2 + (dy)^2} \, dt$$

from our 8.1 explorations, we know that

$$(ds)^2 = (dx)^2 + (dy)^2$$

where $ds = \text{arc length}$

so $ds = \sqrt{(dx)^2 + (dy)^2}$ but $dx \to \frac{dx}{dt}$

$dy \to \frac{dy}{dt}$

$\ y$-axis

$$S = \frac{\pi}{a} \sqrt{(f'(t))^2 + (g'(t))^2} \, dt$$

$$S = \frac{\pi}{a} \int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$
Take a look at this

\[ \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 \]

has some interesting results when we return it to the rectangular coordinate system.

Recall that

\[ \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dx}{dt} \]

\[ \frac{dx}{dt} (dx) = \frac{dy}{dt} \]

\[ \frac{dx}{dt} = \frac{dy}{(dx)} \]

Let's rewrite without parameter t

\[ \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \]

\[ = \sqrt{\left( \frac{dy}{dt} \right)^2 \left( \frac{dx}{dy} \right)^2 + \left( \frac{dy}{dt} \right)^2} \]

\[ = \sqrt{\left( \frac{dy}{dt} \right)^2 \left( \frac{dx}{dy} \right)^2 + 1} \]

\[ = \frac{dy}{dt} \sqrt{\left( \frac{dx}{dy} \right)^2 + 1} \]

Substitute \( \frac{dx}{dt} = \frac{dy}{(dx)} \)

\[ \sqrt{(dx)^2 + (dy)^2} \]

\[ = \sqrt{1 + \left( \frac{dx}{dy} \right)^2} \]

\[ \int \]
If you substitute \( \frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} \) into \( \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \) you get:

\[
\int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
\]

\[
= \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dx} \frac{dx}{dt}\right)^2} \, dt
\]

\[
= \int \sqrt{\left(\frac{dx}{dt}\right)^2 \left(1 + \left(\frac{dy}{dx}\right)^2\right)} \, dt
\]

\[
= \int \frac{dx}{dt} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dt = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx
\]

So, if we look at arc length (and hence area of a surface rotated about an axis) from a parametric equation point of view, you can see why we have the option to make adjustments to our area equations when we are using the cartesian coordinate system.