**Polar Graphing Guide**

**Some hints and observations**

**Basic polar coordinate definitions**

\[
\begin{align*}
x &= r \cos \theta \\
y &= r \sin \theta
\end{align*}
\]

So

\[
\begin{align*}
rcos\theta &= a \\
r \sin \theta &= b
\end{align*}
\]

\[r = \pm a \cos \theta\]

Circle of radius "a" to the right or left of the origin.

\[r = 4 \cos \theta\]

\[2a = 4\\a = 2\text{ (the radius)}\]

\[2a = 6\\a = 3\text{ (the radius)}\]

\[r = \pm a \sin \theta\]

Circle of radius "a" above or below the origin

\[r = -6 \cos \theta\]

**Diagram**

- **Center (0, 0)**
  - Shifts right from 0 to \(\pi\)

- **Center (-3, 0)**
  - Shifts left from 0 to \(\pi\)
\[ r = 2 \sin \theta \]

\[ 2a = 2 \quad \frac{a}{2} = 1 \]
radius.

(center \( (0, 1) \))

from 0 to \( \pi \)

\[ r = -4 \sin \theta \]

\[ 2a = 4 \quad \frac{a}{2} = 2 \]

center \((0, -2)\)

radius

from 0 to \( \pi \)

Limacons \[ r = a \pm b \sin \theta \] / \[ r = a \pm b \cos \theta \]

\[ r = a \pm b \sin \theta \]

or

\[ r = a \pm b \cos \theta \]

\( a/b < 1 \)

\( a/b = 1 \)

\( 1 < a/b < 2 \)

\( a/b \geq 2 \)

Limacon with Inner Loop

Cardioid

Dimpled Limacon

Convex Limacon

Consider:
\[ a = b \rightarrow \text{Cardioid} \]

\[ r = a (1 - \cos \theta) \]
First let's explore what happens when the signs change and the difference between sin and cos graphs

\[ r = 2 + 4 \cos \theta \]

Recall
\[ x = r \cos \theta \]
\[ y = r \sin \theta \]

\[ r = 2 + 4 \cos \theta \]

\[ \frac{2}{4} = \frac{1}{2} < 1 \]
\[ \Rightarrow \text{loop} \]

\[ r = 2 - 4 \cos \theta \]

\[ \frac{2}{-4} = -\frac{1}{2} < 1 \]
\[ \Rightarrow \text{loop} \]
Recall
\[ x = r \cos \theta \\
 y = r \sin \theta \]

\[ r = -2 + 4 \cos \theta \]

\[ r = -2 - 4 \cos \theta \]
Recall:

\[ x = r \cos \theta \]
\[ y = r \sin \theta \]

\[ r = 2 + 4 \sin \theta \]
\[ +y \text{ axis} \]

\[ r = 2 - 4 \sin \theta \]
\[ -y \text{ axis} \]
$$r = -2 + 4 \sin \theta$$

Polar plot:

$$r = -2 - 4 \sin \theta$$

Polar plot:
note that these graphs are the same

\[ r = 2 + 4 \sin \theta \quad \quad r = -2 + 4 \sin \theta \]

so \[ r = 2 + 4 \sin \theta \]

so the sign of a does not affect the axis location
the sign of b does affect the axis location (direction)
more about Limacons

Now let's explore what happens when we adjust $a$ & $b$.

Polar plot: $r = 4 + 2 \sin(\theta)$

- Convex

Polar plot: $r = 3 + 2 \sin(\theta)$

- Dimpled

$\frac{a}{b} > 2$

$\frac{a}{b} = \frac{4}{2} = 2 \geq 2$

Polar plot: $r = 2 + 2 \sin(\theta)$

- Cardiod

$\frac{a}{b} = \frac{3}{2} = 1.5$

Polar plot: $r = 2 + 4 \sin(\theta)$

- Inner loop

$\frac{a}{b} < 1$

$\frac{a}{b} = \frac{2}{2} = 1$

$\frac{a}{b} = \frac{2}{4} = \frac{1}{2}$
Lemniscates (propellers)

\[ r^2 = a^2 \cos 2\theta \quad \text{or} \quad r^2 = -a^2 \cos 2\theta \]
\[ r^2 = a^2 \sin 2\theta \quad \text{or} \quad r^2 = -a^2 \sin 2\theta \]

Due to the \( r^2 \), we have to adjust our logic.

\[ r^2 = 4 \cos 2\theta \]

Plot: \( r^2 = 4 \cos(2\theta) \)

Polar plots:

\[ r^2 = -4 \cos 2\theta \]

Plot: \( r^2 = -4 \cos(2\theta) \)

Polar plots:

\[ + \cos \rightarrow \text{end of propellers on x-axis} \]

\[ - \cos \rightarrow \text{end of propellers on y-axis} \]
\[ r^2 = 4 \sin 2\theta \]

\[ r^2 = -4 \sin 2\theta \]

Polar plots:

1. \( r^2 = 4 \sin 2\theta \)
   - Plot shows an 8-shaped curve.
   - \( \theta \) from \(-\pi\) to \(\pi\).

2. \( r^2 = -4 \sin 2\theta \)
   - Plot shows an 8-shaped curve.
   - \( \theta \) from \(-\pi\) to \(\pi\).

**End of propellers on line \( y = x \)**

**End of propellers on line \( y = -x \)**

\[ \sin \theta = 1 \]
\[ \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots \]
\[ 2\theta = \pi, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots \]
\[ \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \ldots \]

\[ \sin \theta = -1 \]
\[ \Rightarrow \theta = \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots \]
\[ 2\theta = \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots \]
\[ \theta = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \ldots \]

A similar process works for \( \cos(2\theta) \).
Rose Curves

\[ r = \sin(n\theta) \quad \text{or} \quad r = \cos(n\theta) \]

If \( n \) is odd, then the curve has "n" equally spaced petals.
If \( n \) is even, then the curve has "2n" equally spaced petals.

\[ \sin/\text{even} \quad r = 2\sin(6\theta) \]

\[ \sin/\text{odd} \quad r = 2\sin(5\theta) \]

Polar plot:

- \( n = 6 \) (even) \quad 2n \text{ loops} (12)
- \( n = 5 \) (odd) \quad n \text{ loops} (5)
The tips of the petals will be in different locations depending on whether the graph is a \textit{sin} or a \textit{cos} graph.

The example on the next page explores $2\theta$.

The general approach is:

\[ n\theta = A, B, C, \ldots \]
\[ \Theta = \frac{A}{n}, \frac{B}{n}, \frac{C}{n}, \ldots \]
How do sine & cosine graphs differ for Roses?

\[ r = 2 \cos(2\theta) \]

\[ r = 2 \sin(2\theta) \]

**For Cos:**
- Max occurs: \( \theta = 0, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots \)
- Min occurs: \( \theta = \pi, 2\pi, 3\pi, \ldots \)
- Zeros: \( \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots \)

\[ \cos \theta = 1 \]

**For Sin:**
- Max occurs: \( \theta = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots \)
- Min occurs: \( \theta = 3\pi, 7\pi, 11\pi, \ldots \)
- Zeros: \( \theta = 0, \pi, 2\pi, \ldots \)

\[ \sin \theta = 1 \]

We use this information to help us when we are graphing.
Graphing Polar Curves

\[ r \cos \theta = a \rightarrow x = a \rightarrow \text{vertical line} \]
\[ r \sin \theta = b \rightarrow y = b \rightarrow \text{horizontal line} \]
\[ r = a, \text{ is a circle centered at the origin.} \]
\[ \theta = a, \text{ is an angled line.} \]
\[ r = \pm 2a \cos \theta, \text{ is a circle of radius } "a" \text{ right or left of the origin.} \]
\[ r = \pm 2a \sin \theta, \text{ is a circle of radius } "a" \text{ above or below the origin.} \]

Limacons

\[ r = a \pm b \sin \theta \quad \text{or} \quad r = a \pm b \cos \theta \]

\[ \frac{a}{b} < 1 \]
Limacon with Inner Loop

\[ \frac{a}{b} = 1 \]
Cardioid

\[ 1 < \frac{a}{b} < 2 \]
Dimpled Limacon

\[ \frac{a}{b} \geq 2 \]
Convex Limacon

Consider:
\[ a = b \rightarrow \text{Cardioid} \]
\[ r = a (1 - \cos \theta) \]
\[ r^2 = a^2 \cos 2\theta \quad \text{or} \quad r^2 = -a^2 \cos 2\theta \]
\[ r^2 = a^2 \sin 2\theta \quad \text{or} \quad r^2 = -a^2 \sin 2\theta \]

Lemniscate (propeller)

Rose Curves

\[ r = a \sin (n\theta) \quad \text{or} \quad r = a \cos (n\theta) \]

If \( n \) is odd, then the curve has "\( n \)" equally spaced petals.
If \( n \) is even, then the curve has "\( 2n \)" equally spaced petals.
Polar Curves — Match the Graph with the Equation

Polar Equations

A) \( r = -6 \cos \alpha \)

B) \( r = 3 (1 - \sin \alpha) \)

C) \( r = 1 + 2 \sin \alpha \)

D) \( r = 3 - \cos \alpha \)

E) \( r = -3 - 4 \sin \alpha \)

F) \( r^2 = \sin 2\alpha \)

G) \( r = \sin 3\alpha \)

H) \( 2r = \cos \alpha \)

I) \( r = 4 - 4 \cos \alpha \)

J) \( r = 4 + 3 \cos \alpha \)

K) \( r = 5 + 3 \sin \alpha \)

L) \( r^2 = 9 \cos 2\alpha \)

M) \( r = \cos 2\alpha \)

N) \( r = 9 \sin 4\alpha \)

O) \( r = -3 \sin \alpha \)

P) \( r = 2 + 2 \cos \alpha \)

Q) \( r = 1 - 2 \cos \alpha \)

R) \( r = 2 + \sin \alpha \)

S) \( r = 5 - 2 \cos \alpha \)

T) \( r^2 = -16 \sin 2\alpha \)

U) \( r = 2 \cos 3\alpha \)

V) \( r = 1 + \sin \alpha \)

W) \( r = -5 + 5 \sin \alpha \)

X) \( r = 3 + 2 \sin \alpha \)

Y) \( r = 3 + 4 \cos \alpha \)

Z) \( r^2 = -9 \cos 2\alpha \)

AA) \( r = 3 \sin 2\alpha \)

BB) \( r = \cos 5\alpha \)

Answers

1. A

2. O

3. H

4. V

5. B

6. P

7. I

8. W

9. C

10. Q

11. J

12. X

13. D

14. R

15. K

16. Y

17. E

18. S

19. L

20. Z

21. F

22. T

23. M

24. AA

25. G

26. U

27. N

28. BB