Section 5.1/5.3/5.4

5.4

(16) \[ g(x) = \frac{x^3}{x^2-3} \]

Step 1: Find 1st derivative, so use Quotient rule for division

\[ f = x^3 \quad g = x^2 - 3 \]

\[ f' = 3x^2 \quad g' = 2x \]

\[ \frac{f'g - g'f}{g^2} = \frac{3x^2(x^2-3) - 2x(x^3)}{(x^2-3)^2} \]

\[ = \frac{3x^4 - 9x^2 - 2x^4}{(x^2-3)^2} = \frac{\frac{\chi^4-9\chi^2}{(\chi^2-3)^2}}{\frac{\chi^2(\chi^2-9)}{(\chi^2-3)^2}} \]

Easiest format to find 2nd derivative

Step 2: Find 2nd derivative

\[ f = x^4 - 9x^2 \quad g = (x^2-3)^2 \]

\[ f' = 4x^3 - 18x \quad g' = 2(x^2 - 3) \cdot 2x \]

\[ \frac{f'g - g'f}{g^2} = \frac{(4x^3 - 18x)(x^2 - 3)^2 - 4x(x^2 - 3)(x^4 - 9x^2)}{(x^2 - 3)^2} \]

\[ = \frac{(x^2 - 3)^2}{(x^2 - 3)^4} \cdot \frac{(4x^3 - 18x)(x^2 - 3) - 4x(x^4 - 9x^2)}{(x^2 - 3)^2} \]

\[ = [4x^6 - 12x^3 - 18x^3 + 54x - 4x^5 + 36x^2] \text{ FOILED} \]

Then simplify!!
\[
= \frac{4x^5 - 12x^3 - 18x^3 + 54x - 4x^5 + 36x^3}{(x^2 - 3)^3}
\]

Bic outer term was \((x^2 - 3)\) took away 1

Now cancel any from above 66

\[
= \frac{6x^3 + 54x}{(x^2 - 3)^3} \quad J = \frac{6x(x^2 + 9)}{(x^2 - 3)^3}
\]

So

\[
g(x) = \frac{x^3}{x^2 - 3}
\]

\[
g'(x) = \frac{x^4 - 9x^2}{(x^2 - 3)^2} = \frac{x^2(x+3)(x-3)}{(x^2 - 3)^2}
\]

\[
g''(x) = \frac{6x^3 + 54x}{(x^2 - 3)^3} = \frac{6x(x^2 + 9)}{(x^2 - 3)^3}
\]

\[
\text{Step 3: 1st Derivative Analysis}
\]

\[
g'(x) = 0 \quad \text{(Stationary)}
\]

* Bic of denominator we can ignore denominator while finding numerator = 0

\[
\begin{align*}
&\text{SO } x^2(x+3)(x-3) = 0 \\
&x = 0 \quad x + 3 = 0 \quad x - 3 = 0
\end{align*}
\]

\[
\text{3 stationary points (CP)}
\]

\[
g''(x) = \text{undefined} \quad \text{(Singular)}
\]

* Care about setting denominator = 0, so

\[
\begin{align*}
&x^2 - 3 = 0 \\
&x^2 = 3 \\
&x = \pm \sqrt{3}
\end{align*}
\]

\[
\text{1 singular point}
\]
Graph # line

-3, -\sqrt{3}, 0, \sqrt{3}, 3

Test Points:

<table>
<thead>
<tr>
<th>x</th>
<th>g'(x)</th>
<th>Sign</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>6.6</td>
<td>+</td>
</tr>
<tr>
<td>-2</td>
<td>-20</td>
<td>-</td>
</tr>
<tr>
<td>-1</td>
<td>-2</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>-2</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>-20</td>
<td>-</td>
</tr>
<tr>
<td>4</td>
<td>6.6</td>
<td>+</td>
</tr>
</tbody>
</table>

Summary:

CP: -3, -\sqrt{3}, 0, \sqrt{3}, 3

Max = -3 stationary (blc (+) to (-))

Min = 3 stationary (blc (-) to (+))

Step 4: 2nd Derivative Analysis

*Know that the points @ -\sqrt{3}, 0, \sqrt{3} will be included in 2nd derivative analysis, so we can confirm if they are in fact IP (inflection points).
\[
g'' = \frac{6x(x^2+9)}{(x^2-3)^3}
\]

\[
g'' = 0 \quad \text{Num} = 0
\]

\[
6x(x^2+9) = 0
\]

\[
\begin{align*}
x & = 0 \\
(x^2+9) & = 0 \\
-x^2 & = -9 \\
x^2 & = 9 \\
x & = \pm 3
\end{align*}
\]

\[
\chi = 0
\]

\[
\chi = \pm 3i
\]

\[
\text{Note: Imaginary #'s do not appear on a real graph}
\]

\[
g'' \# \text{ line } y_3
\]

\[
\begin{cases}
g''(-2) = -156 \quad (-) \\
g''(-1) = -17.5 \quad (+) \\
g''(1) = -7.5 \quad (-) \\
g''(2) = 156 \quad (+)
\end{cases}
\]

\[
\text{Potential inflection points}
\]

\[
\text{Summary:}
\]

\[
\text{Potential IP} \Rightarrow -\sqrt{3}, 0, \sqrt{3}
\]

\[
\text{Note: do not need to know or note about stationary/singular when dealing w/ 2nd derivative.}
\]
STEP 5: Before we have a denominator in \( g(x) \) we have additional things to check.

1st: Check vertical asymptotes (VA)

Where denominator of original \( g(x) = 0 \)

So \( g(x) = \frac{x^3}{x^2-3} \), take denominator

\[ x^2 - 3 = 0 \]

\[ x^2 = 3 \]

\[ x = \pm \sqrt{3} \]

VA

2nd: Check horizontal asymptotes (HA)

What we are actually checking

\[ \lim_{x \to \infty} f(x) = ax^n + \ldots \]

\[ = \frac{5x^3}{x^3} + \ldots \]

So if \( n > d \) = no HA, but other lines may exist.

**Remember rule**

\[ n = d \] so \( y = \frac{a}{b} \)

\[ n < d \] \( y = 0 \)

So

\[ \frac{x^3}{x^2-3} \]

\[ \frac{x^3}{x^2} = x \]

\[ \frac{\text{blc} \ x^3 > x^2}{\text{we know there is no HA.}} \]

Line looks like this
Use long division to solve for oblique asymptote:

\[
x^2 - 3 \div x^3 + 0x^2 + 0x + 0
\]

\[
\begin{array}{c|cccc}
  & x^3 & +0x^2 & +0x & +0 \\
\hline
  & x^3 & +3x^2 & & \\
  \hline
  & 0x^2 & +0x & +0 \\
  & 0x^2 & +0x & +0 \\
  \hline
  & 0x^2 & +0x & +0 \\
\end{array}
\]

\[
\frac{3x+0}{x^2 - 3} = \frac{3x}{x^2 + 3}
\]

Redo with remainder

\[
\frac{3x+0}{x^2 - 3}
\]

Equation for asymptote:

\[y = x\]

**Step 6** Set points plotted on graph using g(x):

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>-4.5</td>
</tr>
<tr>
<td>-\sqrt{3}</td>
<td>\sqrt{4} or DNE</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>\sqrt{3}</td>
<td>\sqrt{4} or DNE</td>
</tr>
</tbody>
</table>

**If any addl. points add to table, not test points**
Graph

Put asymptotes

\[ y = x \]

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>