The Big Idea:

**Theorem** Let $F$ be a fixed point (called the focus) and $l$ be a fixed line (called the directrix) in a plane. Let $e$ be a fixed positive number (called the eccentricity). Then the set of all points $P$ in the plane such that

$$\frac{|PF|}{|PL|} = e$$

is a conic section.

Specifically,

a) an ellipse if $e < 1$,

b) a parabola if $e = 1$

c) a hyperbola if $e > 1$

converted format of equations for conic sections

$$r = \frac{ed}{1 \pm e \cos \theta} \quad \text{or} \quad r = \frac{ed}{1 \pm esin \theta}$$

$\cos \theta \rightarrow$ symmetry WRT to $x$

$\sin \theta \rightarrow$ symmetry WRT $y$

$d = \text{directrix}$

recall from parabolas
directrix $y = -p$ or $x = -p$
Visuals

$x^2 = 4py$ $(p > 0)$

![Graph of $x^2 = 4py$ with focus at $(0, p)$ and directrix $y = -p$.]

$y^2 = 4px$ $(p > 0)$

![Graph of $y^2 = 4px$ with focus at $(p, 0)$ and directrix $x = -p$.]

Other useful information

$e < 1$ ellipse

$c^2 = a^2 - b^2$

$e = \frac{c}{a}$

$e \geq 1$ hyperbola

$c^2 = a^2 + b^2$

$e = \frac{c}{a}$

the directrix is the same distance (opposite direction) of the focus point.
Example 1

Find a polar equation for a parabola that has a focus at the origin and whose directrix is the line \( y = -6 \)

- **Parabola**: \( e = 1 \)
- **Focus**: \( (0,0) \)
- **Directrix**: \( y = -6 \) \( \Rightarrow d = 6 \)

![Diagram of parabola with focus and directrix]

We choose (from our 4 options on page 684)

\[
r = \frac{ed}{1 - esin\Theta}
\]

\[
r = \frac{(1)(6)}{1 - (1)sin\Theta}
\]

\[
r = \frac{6}{1 - sin\Theta}
\]
Example 2

Describe and sketch a graph of the equation

\[ r = \frac{10}{3 - 2 \cos \theta} \]

First, we need to match one of the formats above

\[ r = \frac{10}{3 - \frac{2}{3} \cos \theta} \]

(form b)

Since \( \frac{2}{3} < 1 \) \Rightarrow ellipse

With focus F

and major axis along the polar axis.

Major axis \( \Rightarrow x \) since \( d = -x \)
To find the endpoints of the major axis, set $\theta = 0$ and $\theta = \pi$.

**$\theta = 0$**

\[ r = \frac{10}{3 - 2 \cos(0)} = \frac{10}{3} = 10 \quad (10, 0) = (r, \theta) \]

**$\theta = \pi$**

\[ r = \frac{10}{3 - 2 \cos(\pi)} = \frac{10}{3 + 2} = 2 \quad (2, \pi) = (r, \theta) \]

The center = midpoint of $(2, 0)$ & $(10, \pi)$

\[ \Rightarrow \text{distance} = 6 \quad \text{or} \quad a = 6 \]

basic sketch

The center is the midpoint of $(2, 0)$ and $(10, \pi)$, so the distance between them is $6$, which means $a = 6$.

We know $e = \frac{|\text{dF}|}{|\text{dl}|}$

\[ e = \frac{2}{3} = \frac{c}{a} \]

So

\[ c^2 = a^2 - b^2 \]

\[ 4^2 = 6^2 - b^2 \]

\[ 16 = 36 - b^2 \]

\[ -20 = -b^2 \]

\[ \sqrt{20} = b \]

Length of minor axis is $2c$

\[ \frac{2}{3} = \frac{c}{6} \]

\[ c = 4 \]
Example 3

Describe and sketch the graph of the equation

\[ r = \frac{10}{\frac{a}{2} + 3\sin\theta} \]

\[ r = \frac{10}{\frac{a}{2}} = \frac{5}{\frac{1 + \frac{3}{2}\sin\theta}{3}} \]

\[ y = d \]

match \[ \frac{ed}{1 + esin\theta} \]

ed = 5 \quad e = \frac{3}{a} \quad \left(\frac{3}{2}\right)d = 5

\[ d = \frac{5(2)}{3} = \frac{10}{3} \]

\[ e = \frac{3}{a} \]

\[ e > 0 \]

\[ \text{hyperbola} \]

what do we need to know?

\[ \theta = \frac{\pi}{2} \quad r = \frac{5}{1 + \frac{3}{2}\sin\left(\frac{\pi}{2}\right)} = 2 \]

\[ \theta = \frac{3\pi}{2} \quad r = \frac{5}{1 + \frac{3}{2}\sin\left(\frac{3\pi}{2}\right)} = -10 \]

center, a, b

asymptotes endpoints

endpoints

\( (a, \frac{\pi}{2}) \)

\( (-10, \frac{3\pi}{2}) \)
**Summary of info, so far**

\[ e = \frac{3}{2} > 0 \quad \text{hyperbola with focus at the poles.} \]

\[ \sin \theta \rightarrow \text{transverse axis of the hyperbola is } \perp \text{ to the polar axis.} \]

Vertices: \((2, \frac{11}{2})\) \((10, \frac{3\pi}{2})\)

The graph so far

\[
\begin{align*}
2a &= 8 \\
\Rightarrow a &= 4 \\
c &= 6 \\
(2, \frac{\pi}{2})
\end{align*}
\]

When \(e > 1\)

\[ e = \frac{c}{a} \]

\[ c^2 = a^2 + b^2 \]

\[
\begin{align*}
\frac{3}{2} &= \frac{c}{4} \\
12 &= c = 6 \\
\frac{b^2}{a^2} &= \frac{36}{16} \\
b^2 &= 36 - 16 \\
b &= \sqrt{20}
\end{align*}
\]

Asymptotes \(y = \pm \frac{a}{b}x\)
Example 4

Sketch the graph of the equation

\[ r = \frac{15}{4 - 4\cos \theta} \]

\[ r = \frac{15}{4} \frac{1}{\cos \theta - \frac{4}{4}} \]

\[ e = 1 \]

parabola with focus point at the pole

For parabolas, you can find a good rough sketch by plotting a few points

\[
\begin{array}{c|cccc}
\theta & 0 & \frac{\pi}{2} & \pi & \frac{3\pi}{2} \\
r & \text{und} & 15/4 & 15/8 & 15/4 \\
& (3 3/4) & (1 7/8) & (3 3/4)
\end{array}
\]